Math 33A Worksheet Week 9

TA: Emil Geisler and Caleb Partin

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Exercise 1.
Recall that we can compute the determinant of a $n \times n$ matrix through **Cofactor Expansion**. If A is the matrix we are trying to compute the determinant of, then we can let A_{ij} denote the $(n-1) \times (n-1)$ matrix formed by removing the ith row and jth column from A. Then for some choice of column j

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}$$

or some choice of row i

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

That is, we can compute the determinant by summing over the cofactors for any row and column. Use these formulas to compute the determinant of the following matrices:

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & -1 & 3 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 \end{bmatrix}$

Exercise 2. One important application of determinants is that they help us determine invertibility of a matrix. Use the determinant to determine for which values λ the following matrices are invertible:

(a) $\begin{bmatrix} \lambda & 2 \\ 3 & 4 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 1 & \lambda \\ 1 & \lambda & \lambda \\ \lambda & \lambda & \lambda \end{bmatrix}$

(c) $\begin{bmatrix} 1-\lambda & 2 \\ 0 & 4-\lambda \end{bmatrix}$

(d) $\begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 1 \\ 0 & -1 & 6-\lambda \end{bmatrix}$

Exercise 3. Determine whether the following are true or false:

(a) $\det(A + B) = \det(A) + \det(B)$ for any two $n \times n$ matrices A and B.

(b) If B is the rref of A, then $\det(B) = \det(A)$.

(c) There exists a 3×3 matrix A with real valued entries such that $A^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(d) If A is an orthogonal matrix, then $\det(A) = \pm 1$.

Exercise 4. Write down the relationship between $\det(B)$ and $\det(A)$ for the following scenarios (**Challenge:** Can you prove any or all of them?)

(a) If B is obtained by multiplying a row of A by a scalar k.

(b) If B is obtained by swapping two rows of A.

(c) If B is obtained by adding a multiple of one row of A to another row of A.

(d) $B = A^T$

(e) If A is invertible, $B = A^{-1}$
